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Strength coefficients of the cubic criterion for graphite composites

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Based upon the strength criterion proposed by Gol'denblat and Kopnov, the second-, fourth-, and sixth-rank strength tensors in the three-dimensional case have been determined for each of the crystal classes from consideration of invariant transformations of the strength function. A cubic polynomial form has been proposed as an improved strength criterion for anisotropic brittle materials by using the constitutive laws in continuum mechanics. The methods of determination of the strength coefficients and the strength envelope in the biaxial stress state for graphite are presented in detail by relating the cubic polynomial strength criterion to the experimental test data. Numerical examples are presented to show that the theoretical and experimental results are in good agreement. A comparison with the results previously published is presented.

1. Introduction

Graphite is a non-linear material. Many grades of graphite are readily available and they have many useful properties for special applications. Graphite has a relatively high thermal conductivity, low thermal expansion coefficient, high resistance to thermal shock, high melting point, and great resistance to chemical erosion. By virtue of its high strength-to-weight ratio, graphite has become one of the leading materials to be used in rockets and missiles as irregular cone, nozzle, and vane shapes. In the nuclear industry, graphite is used as a moderator, a reflector or a thermal column, and a shielding structure because of the increase in strength and hardness of graphite as it is exposed to irradiation.

Prior to fracture, graphite exhibits a small amount of plastic deformation. Like most other brittle materials, graphite is compressible and has a higher strength in compression than in tension. This is commonly referred to as the Bauschinger effect.

Graphite has a material symmetry unlike many other brittle materials. The plane perpendicular to the direction of extrusion is considered to be an isotropic plane. The material properties in the isotropic plane may be quite different from the properties along the direction of extrusion. The material symmetry so described is called transverse isotropy.

The broad usage of graphite demonstrates that it is indeed a versatile industrial material. Thus, for the purpose of material characterization and design, an operationally simple strength criterion for graphite is desirable.

In recent years the trend has been to develop a strength criterion based on the invariants of the stress tensor in a unified form. This form of higher order function is more suited to incorporate with the computer codes and enables the inclusion of more stress interaction terms yielding an accurate description of a failure surface.

Based upon the strength criterion proposed by Gol'denblat and Kopnov, Huang [1] determined the second-; fourth-; and sixth-rank strength tensors in the three-dimensional case for each of the crystal classes from consideration of invariant transformations of the strength function. Huang [2] proposed a cubic polynomial form as an improved strength criterion for anisotropic brittle materials using the constitutive laws in continuum mechanics. In the present paper the methods of determination of the strength coefficients and the strength envelope in the biaxial stress state for graphite are presented in detail, by relating the cubic polynomial strength criterion to the experimental test data. A numerical example is presented to show that the theoretical and experimental results are in good agreement. A comparison with previous results published [4] is presented.

2. Development of strength criterion

The general theory of strength criterion for anisotropic crystals can be established from consideration of the strength functions F, where

$$F = F(\sigma_{ij}) = 0 \qquad (1)$$

and σ_{ij} is the stress tensor which is symmetric. The strength function, F, is required to be invariant under the group transformations $\{t_{ij}\}$ which characterize the material anisotropy

$$F(\bar{\sigma}_{ij}) = F(\sigma_{ij}) \qquad (2)$$

where the transformed stress tensor obeys the following rule

$$\bar{\sigma}_{ij} = t_{ir} t_{js} \sigma_{rs}$$

It is also assumed that the strength function F may be expressed approximately in the following polynomial form as proposed by Col'denblat and Kopnov [3]

$$F = (F_{ij}\sigma_{ij})^{\alpha} + (F_{ijkl}\sigma_{ij}\sigma_{kl})^{\beta} + (F_{ijklmn}\sigma_{ij}\sigma_{kl}\sigma_{mn})^{\gamma} - 1 = 0 \qquad (3)$$

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which, in fact, is a third-order approximation of Equation 1.

For the transversely isotropic material such as graphite we take, as our reference system, a right-hand rectangular cartesian coordinate system X_i, such that the X_3 is parallel to the direction of material extrusion. The group of coordinate transformation is

$$R_\alpha[\bar{x}_1 + i\bar{x}_2 = e^{-i\alpha}(x_1 + ix_2),\ \bar{x}_3 = x_3] \quad (4)$$

for all values of α, and the stress invariants are

$$I^{(1)}: \sigma_3,\ \sigma_1 + \sigma_2$$
$$I^{(2)}: \sigma_4^2 + \sigma_5^2,\ \sigma_1\sigma_2 - \sigma_6^2 \quad (5)$$
$$I^{(3)}: \{\det|\sigma_i|\} \text{ or } 2\sigma_4\sigma_5\sigma_6 - \sigma_1\sigma_4^2 - \sigma_2\sigma_5^2$$

where the contracted notation is used, i.e.

$$(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12}) = (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6)$$

The stress invariants presented in Equation 5 form a strength function F in order for the transversely isotropic hyperelastic materials. With Equation 5 the cubic strength function F can be written as the following form-invariant

$$\begin{aligned}&\{F_1(\sigma_1 + \sigma_2) + F_3\sigma_3\}^\alpha + \{F_{11}(\sigma_1^2 + \sigma_2^2) + F_{33}\sigma_3^2\\ &+ 2F_{13}[\sigma_3(\sigma_1 + \sigma_2)] + 2F_{12}\sigma_1\sigma_2 + F_{55}(\sigma_4^2 + \sigma_5^2)\\ &+ 2(F_{11} - F_{12})\sigma_6^2\}^\beta + \{F_{333}\sigma_3^3 + F_{111}(\sigma_1 + \sigma_2)^3\\ &+ F_{113}(\sigma_1 + \sigma_2)^2\sigma_3 + F_{133}(\sigma_1 + \sigma_2)\sigma_3^2\\ &+ F_{316}[(\sigma_1\sigma_2 - \sigma_6^2)\sigma_3] + F_{355}[(\sigma_4^2 + \sigma_5^2)\sigma_3]\\ &+ F_{116}[(\sigma_1 + \sigma_2)(\sigma_1\sigma_2 - \sigma_6^2)]\\ &+ F_{155}[(\sigma_1 + \sigma_2)(\sigma_4^2 + \sigma_5^2)]\\ &+ F_{666}[\det(\sigma_i)]\}^\gamma = 1 \quad (6)\end{aligned}$$

where F_i, F_{ij}, and F_{ikj} are the coefficients of strength in various orders, σ_i ($i = 1, 2, \ldots, 6$) is the matrix for the stress components, and α, β, and γ are material parameters.

For the plane problem in (X_1, X_3), Equation 6 yields

$$\begin{aligned}&(F_1\sigma_1 + F_3\sigma_3)^\alpha + (F_{11}\sigma_1^2 + F_{33}\sigma_3^2 + 2F_{13}\sigma_1\sigma_3\\ &+ F_{55}\sigma_5^2)^\beta + (F_{111}\sigma_1^3 + F_{333}\sigma_3^3 + F_{113}\sigma_1^2\sigma_3\\ &+ F_{133}\sigma_1\sigma_3^2 + F_{355}\sigma_3\sigma_5^2 + F_{155}\sigma_1\sigma_5^2)^\gamma = 1 \quad (7)\end{aligned}$$

The function given in Equation 6 or Equation 7 is invariant under the group of transformations of coordinates given in Equation 4. For the case of $\alpha = \beta = \gamma = 1$, the tensors F_i and F_{ijk} characterize the Bauschinger effect of the material, and the tensors F_{ij} and F_{ijk} determine the hypersurface of the strength function in the stress space. It is apparent that the third-order approximations contain more disposable coefficients, as compared with the quadratic approximation of strength tensor theory. Therefore, the latter form is more flexible and improved the curve fitting. Furthermore, the cubic form of the strength criterion in the X_1X_3 plane with the shear stress being zero can be written as

$$\begin{aligned}F(\sigma_i) &= F_1\sigma_1 + F_3\sigma_3 + F_{11}\sigma_1^2 + 2F_{13}\sigma_1\sigma_3 + F_{33}\sigma_3^2\\ &+ F_{111}\sigma_1^3 + F_{113}\sigma_1^2\sigma_3 + F_{133}\sigma_1\sigma_3^2\\ &+ F_{333}\sigma_3^3 - 1 = 0 \quad (8)\end{aligned}$$

where the strength coefficients F_{13}, F_{113}, and F_{133} characterize completely the interactions of the principal normal stress σ_1 and σ_3 in the graphites. Certainly these coefficients can be determined only by experimental tests of the multiaxial stress states.

3. Determination of strength coefficients

In order to determine the nine components of the strength tensor in Equation 8, we need at least nine independent equations with the strength coefficients as unknowns. In other words, we need tests of nine different stress states. By inspection of Equation 8 it is apparent that nine data sets from biaxial stress tests would give nine simultaneous equations, and hence the nine strength coefficients could be obtained by solving these equations. Although it is possible to get the best selection of data by choosing certain test points with a trial-and-error process, we would like to make use of the whole set of data points. Since there are unavoidable errors existing in each test procedure, it is necessary to have the errors rounded over the whole range of the stress space. In this case, there will be more than nine equations to determine the nine strength coefficients. In order to obtain appropriate values of coefficients, the overdetermined set of simultaneous equations can be solved by using a numerical method such as the least square technique. For n data points, Equation 8 can be written as

$$\begin{aligned}&F_1(\sigma_1)_i + F_3(\sigma_3)_i + F_{11}(\sigma_1^2)_i + F_{13}(2\sigma_1\sigma_3)_i\\ &+ F_{33}(\sigma_3^2)_i + F_{111}(\sigma_1^3)_i + F_{113}(\sigma_1^2\sigma_3)_i\\ &+ F_{133}(\sigma_1\sigma_3^2)_i + F_{333}(\sigma_3^3)_i - 1 = 0 \quad (9)\end{aligned}$$

where $i = 1, 2, \ldots, n$. The matrix form of above equations is expressed as

$$\underset{(n \times 1)}{\{1\}} = \underset{(n \times 9)}{[\sigma]}\ \underset{(9 \times 1)}{\{F\}}$$

where $\{1\}$ denotes the column matrix with all elements being one. To assign appropriate weights to the data points, a diagonal weighting matrix $[w]$ is introduced. By premultiplying Equation (9) with $[\sigma]^T[w]$, one obtains

$$\underset{(9 \times n)}{[\sigma]^T}\ \underset{(n \times n)}{[w]}\ \underset{(n \times 1)}{\{1\}} = \underset{(9 \times n)}{[\sigma]^T}\ \underset{(n \times n)}{[w]}\ \underset{(n \times 9)}{[\sigma]}\ \underset{(9 \times 1)}{\{F\}} \quad (10)$$

when equal weights are given to all data points, $[w]$ is an identity matrix.

Equation 10 can be written as

$$\underset{(9 \times 1)}{\{E\}} = \underset{(9 \times 9)}{[D]}\ \underset{(9 \times 1)}{\{F\}} \quad (11)$$

or

$$\{F\} = [D]^{-1}\{E\}.$$

By solving Equation (11), the nine components of strength tensors F are obtained. Substitution of these computed values of F_1, F_3, . . . , F_{333} in Equation 8 yields the best approximation by the least square fit to the experimental stress data.

4. Determination of the strength envelope for biaxial states

Equation 1 defines geometrically a strength surface in six-dimensional stress space. A stress point can only be located either on or inside the strength surface in the stress space. Failure occurs only if the stress point is on the surface. With the geometric interpretation, Equation 8 gives the strength envelope of the biaxial states of stress. This envelope represents the intersection of the strength surface with the X_1X_3 coordinate plane.

The strength envelope of the biaxial states of stress has to be closed and convex to ensure the stability of the material. In other words, any radial line from the origin of coordinates (zero stress state) must intersect the stress envelope at only two points which are located on opposite sides of the origin. In this paper, the cubic strength function is considered. A cubic strength function with real coefficients (as Equation 8) has three roots in general. However, in order to have a closed envelope enclosing the origin of the stress space, Equation 8 must have three real roots on a loading path. For this reason adequate weights must be assigned to those data points while the strength coefficients, F's, are determined, so that the strength envelope is a closed, smooth, and convex curve.

In this paper, Equation 8 will not be solved analytically in a closed form solution. Alternatively, an iterative procedure of numerical analysis is developed by using the Newton–Raphson technique. Let R denote the ratio of σ_3 to σ_1, or $R = \sigma_3/\sigma_1 = \tan\theta$, where θ represents the slope of a radial line.

For an iteration scheme, Equation 8 can be defined as follows

$$F(\sigma_i) = f$$

where f denotes the residual function. This equation can be expressed as

$$f = A\sigma_1^3 + B\sigma_1^2 + C\sigma_1 - 1 \qquad (12)$$

where

$$A = F_{333}R^3 + F_{133}R^2 + F_{113}R + F_{111}$$
$$B = F_{33}R^2 + 2F_{13}R + F_{11}$$
$$C = F_3R + F_1$$

Of course, if the correct value of σ_1 for a given value of R is substituted into Equation 12, the function f will be equal to zero. On the other hand, if an incorrect value of σ_1 is used, Equation 12 yields a non-zero value.

In order to iterate from an estimated value of σ_1 to a correct solution of σ_1, a Newton–Raphson technique is used. The correction factor

$$\Delta\sigma_1 = -[f/(3A\sigma_1^2 + 2B\sigma_1 + C)] \qquad (13)$$

is determined to improve the previous value of σ_1. Therefore, the improved value of σ_1 is

$$(\sigma_1)_{n+1} = (\sigma_1)_n + (\Delta\sigma_1)_n$$

where n denotes the number of iterative steps.

To start the iteration procedure, an initial guess is required for the unknown, σ_1. This can be done by simply setting σ_1 equal to zero or the uniaxial material strength for $R = 0$. When R is increased by a small increment ΔR, the previously obtained value of σ_1, which is the correct solution associated with previous R value, is used as the initial guess value for the current calculation. This is a particularly appropriate scheme for the strength envelope; the correct solution $(\sigma_1, \sigma_3)_{R=(n+1)\Delta R}$ for the current calculation is in the neighbourhood of the previous solution $(\sigma_1, \sigma_3)_{R=n\Delta R}$. Also this scheme converges quadratically.

As the technique converges to the correct solution, two things happen. The absolute value of the residual function given by Equation 12 approaches zero, and the absolute value of the correction factor given by Equation 13 becomes smaller. Hence, an appropriate small value of these quantities may be chosen as the criterion for which the iterative process is terminated.

In this numerical analysis, a particular point should be noted. In order to obtain a smooth, closed, convex strength envelope, Equation 8 must yield three real roots. After we obtain one real root of Equation 8, the other two roots must be real, also. By eliminating the obtained real root, denoted by σ, one finds a quadratic equation

$$A_1\sigma_1^2 + B_1\sigma_1 + C_1 = 0 \qquad (14)$$

where

$$A_1 = A$$
$$B_1 = B + A_1\sigma$$
$$C_1 = C + B_1\sigma$$

If coefficients A_1, B_1, and C_1 satisfy the inequality

$$B_1^2 - 4A_1C_1 \geqslant 0$$

the two remaining roots of Equation 8 will be real. Otherwise the other two roots are conjugate complex. If the latter case occurs, the strength coefficients F_i, F_{ij} and F_{ijk} must be re-evaluated. However, the iterative process adopted in this paper usually proceeds

TABLE I

	ATJ-S	ATJ	Graph-I-tite	JTA
F_1	0.148 300	0.060 00	0.172 000	0.059 600
F_3	0.147 700	0.210 00	0.128 000	0.165 000
F_{11}	−0.012 700	0.045 00	0.046 900	0.005 000
F_{13}	−0.013 400	−0.007 00	−0.022 000	−0.006 780
F_{33}	0.022 700	0.038 00	0.021 000	0.011 700
F_{111}	−0.000 697	0.002 10	−0.001 600	0.000 040
F_{113}	−0.000 647	−0.006 00	0.001 860	−0.000 300
F_{133}	−0.002 170	0.000 38	−0.003 570	−0.000 070
F_{333}	0.000 445	0.000 30	0.000 082	0.000 128

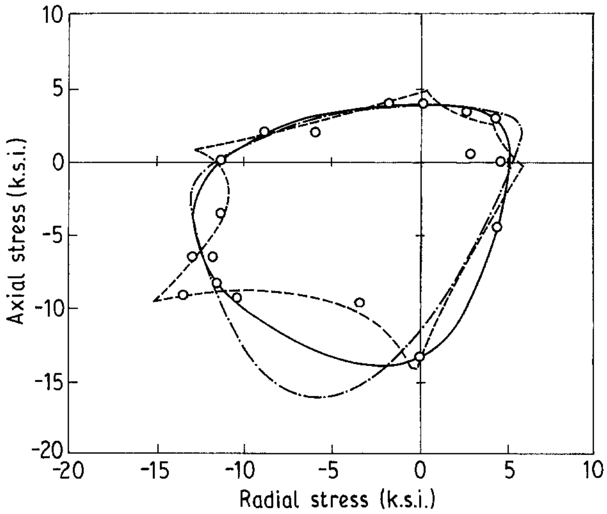

Figure 1 Strength envelopes for ATJ-S graphite (○, data, —— proposed cubic function, - - - higher order failure surface, –·–·–·– non-homogeneous multisegmented ellipse).

smoothly at every stage if R is incremented gradually. The computation results in a desired real root for the true strength envelope. Thus the re-examination of the other two roots to be real is not necessary in this iterative process if the strength coefficients are determined properly. As the process converges to the correct solution, σ_1, for a given value of R, the corresponding value of σ_3 will be obtained easily. In this way the complete strength envelope, as expected, can be obtained graphically.

5. Numerical example

A computer program has been developed. Incorporating the determination of the strength coefficients and the real roots of the cubic equation, the best fit strength envelope is obtained. In order to verify the proposed strength criterion, the experimental data for graphites [4] are used for the forenamed calculations.

For the case of plane stress in the X_1X_3 plane with shear stress σ_5 being zero, the strength coefficients are obtained and summarized in Table I.

The initial value σ_1 is chosen to be zero while

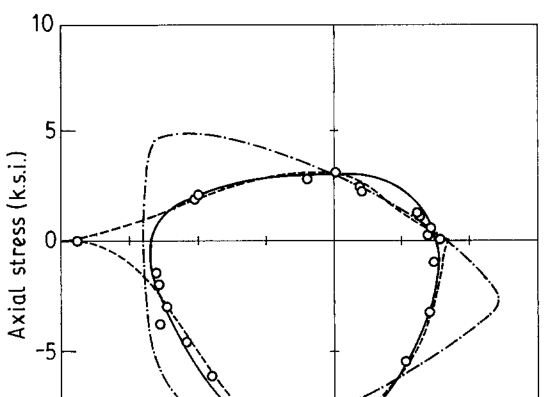

Figure 2 Strength envelopes for ATJ graphite (○, data, —— proposed cubic function, - - - higher order failure surface, –·–·–·– non-homogeneous multisegmented ellipse).

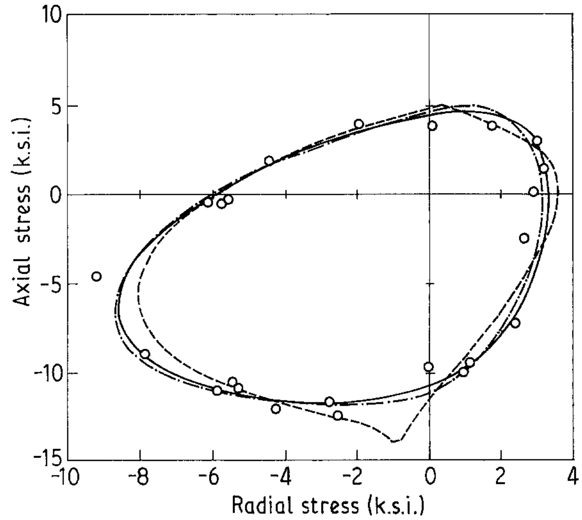

Figure 3 Strength envelopes for graphi-I-tite graphite (○, data, —— proposed cubic function, - - - higher order failure surface, –·–·–·– non-homogeneous multisegmented ellipse).

$R = 0$. In the iterative process, a five degree increment of $\Delta\theta$ is specified, as θ is increased gradually from 0° to 360° ($R = \tan\theta$). When $\theta = 90°$ and 270° a modification must be made in this iterative process in order to avoid the difficulty in numerical calculation. In this calculation, an error tolerance of 1×10^{-4} is used to decide when the iteration should be stopped.

The stress envelope given by Equation 8 with the strength coefficients shown in Table I for graphites are plotted in Figs 1 to 4. The results previously obtained [4] are also plotted in the same figures for comparison.

6. Conclusion

The new proposed cubic polynomial strength criterion (Equation 8) has been compared with the existing experimental strength data for the biaxial stress state. Good correlation between theoretical and experimental results is observed.

Systematic methods of determining strength coefficients as well as the stress envelope have been developed. These methods are also suitable for evaluation of strength criteria for other brittle materials based on the concept of the stress tensor invariants.

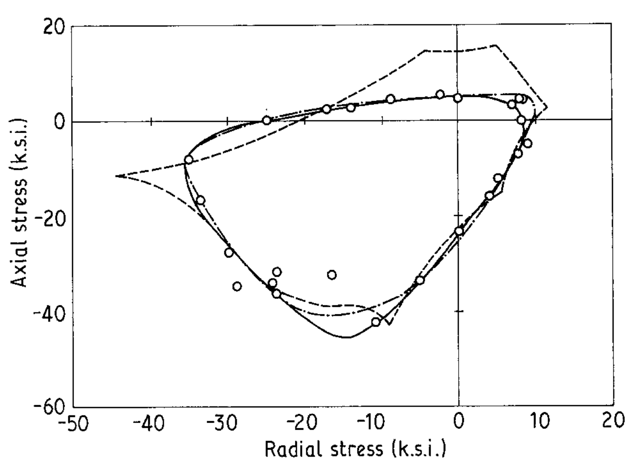

Figure 4 Strength envelopes for JTA graphite (○, data, —— proposed cubic function, - - - higher order failure surface, –·–·–·– non-homogeneous multisegmented ellipse).

The configuration of the stress envelope obtained in this paper is very similar to that previously given [4]. However, in this paper a straightforward approach is taken without using the absolute value term in the strength function, as has been proposed previously [4], (the so-called non-homogeneous multisegmented ellipse). The multisegmented ellipse approach might cause some difficulties in the numerical calculation. On the other hand, a simple cubic order failure surface can yield good results for JTA graphite as it did in this paper. This result also shows that the cubic order failure surface criterion should not be abandoned as has been previously argued [4].

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